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LETTER TO THE EDITOR

On the quantum mechanics of complex Hamiltonian systems in one dimension

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Abstract
The solutions of an analogous Schrödinger wave equation for the one-dimensional non-Hermitian Hamiltonian $H(x, p)$ in the complex phase plane characterized by $x = x_1 + ip_2$, $p = p_1 + ix_2$, are investigated. The quasi-exact solutions thus obtained reveal a lot about the nature of the complex eigenvalue spectrum of the potential concerned. The examples of harmonic, harmonic plus inverse harmonic and Mörse potentials are discussed.

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From the point of view of having a better theoretical understanding of several newly discovered [1,2] phenomena in different branches of science, the study of complex Hamiltonian systems in one space dimension has become of considerable interest [2–11] in recent years. In this context both classical and quantum aspects of the system have been investigated. The fact that a non-Hermitian (particularly the $\mathbb{P}\mathcal{J}$-symmetric one) Hamiltonian $H(x, p)$ can also provide real eigenvalues in a certain parametric domain of the system has been the basis of a number [3–6] of recent studies of the quantum mechanics of complex Hamiltonians. At the classical level, on the other hand, while the construction of exact invariants of such systems has been carried out [9] using a more general transformation [10] for the canonical variables x and p, namely

$$x = x_1 + ip_2 \qquad p = p_1 + ix_2 \tag{1}$$

the analyticity property of the Hamiltonian

$$H(x, p) = H_1(x_1, p_1, x_2, p_2) + iH_2(x_1, p_1, x_2, p_2) \tag{2}$$

has further yielded [11] a class of integrable systems in the associated two space dimensions characterized by the phase space variables (x_1, p_1) and (x_2, p_2).

As a matter of fact, the $\mathbb{P}\mathcal{J}$ symmetry of the non-Hermitian Hamiltonian $H(x, p)$ appears to be a special case of the general transformation (1) (in the sense that under

$\mathbb{P}\mathcal{J}$ symmetry this transformation reduces to a restriction on the variables x_1, p_1, x_2, p_2, such that $(x_1, p_1, x_2, p_2) \rightarrow (-x_1, p_1, -x_2, p_2, \mathrm{i} \rightarrow -\mathrm{i})$. In the present letter we make use of the transformation (1) to study the quantum aspects of a system. In particular, we shall look for the so-called quasi-exact solution of the Schrödinger-like equation

$$\hat{H}(x, p)\psi(x) = E\psi(x) \tag{3}$$

where ($\hbar = m = 1$)

$$\hat{H}(x, p) = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x). \tag{4}$$

As equation (3) departs from the conventional conceptual and mathematical setting of the standard [12] Schrödinger equation, we regard this equation as the *analogous Schrödinger equation* (ASE) for the non-Hermitian operator $\hat{H}(x, p)$. Further, we do not consider (mainly for the sake of convenience) the dependence of ψ on the real time variable t and look here only for the stationary state solutions of equation (3). Note that, in the quantum case, the analyticity property of $\hat{H}(x, p)$—an operator function of two complex variables— translates into the analyticity of the potential function $V(x)$. Even this latter property is not needed unless the formalism itself involves the derivatives of the potential function $V(x)$. However, the analyticity property of $\psi(x)$ will be used in the following work.

Before proceeding further, note from (1) that

$$\frac{\mathrm{d}}{\mathrm{d}x} = \frac{\partial}{\partial x_1} - \mathrm{i}\frac{\partial}{\partial p_2} \qquad \frac{\mathrm{d}}{\mathrm{d}p} = \frac{\partial}{\partial p_1} - \mathrm{i}\frac{\partial}{\partial x_2}. \tag{5}$$

The important postulate of conventional quantum mechanics, namely $p \rightarrow -\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}x}$, now takes the form $(p_1 + \mathrm{i}x_2) \rightarrow -\mathrm{i}\left(\frac{\partial}{\partial x_1} - \mathrm{i}\frac{\partial}{\partial p_2}\right)$, which implies $p_1 \rightarrow -\frac{\partial}{\partial x_2}, x_2 \rightarrow -\frac{\partial}{\partial x_1}$. Next we write the complex quantities $V(x)$, $\psi(x)$ and E in the form

$$V(x) = V_r(x_1, p_2) + \mathrm{i}V_i(x_1, p_2) \qquad \psi(x) = \psi_r(x_1, p_2) + \mathrm{i}\psi_i(x_1, p_2) \qquad E = E_r + \mathrm{i}E_i$$

where the subscripts r and i, respectively, denote the real and imaginary parts of the corresponding quantity. Additional subscripts to these quantities separated by a comma will, however, denote the partial derivatives of the quantity concerned. Thus, using (5) in (4) the ASE (3), when separated into real and imaginary parts, gives rise to a pair of coupled partial differential equations

$$-\frac{1}{2}(\psi_{r,x_1x_1} - \psi_{r,p_2p_2} + 2\psi_{i,x_1p_2}) + V_r\psi_r - V_i\psi_i = E_r\psi_r - E_i\psi_i \tag{6a}$$

$$-\frac{1}{2}(\psi_{i,x_1x_1} - \psi_{i,p_2p_2} - 2\psi_{r,x_1p_2}) + V_i\psi_r + V_r\psi_i = E_r\psi_i + E_i\psi_r. \tag{6b}$$

The use of the analyticity property of $\psi(x)$ in terms of the Cauchy–Riemann conditions, namely

$$\psi_{r,x_1} = \psi_{i,p_2} \qquad \psi_{r,p_2} = -\psi_{i,x_1} \tag{7}$$

will reduce equations (6a) and (6b) to somewhat simpler forms as

$$-2\psi_{r,x_1x_1} + V_r\psi_r - V_i\psi_i = E_r\psi_r - E_i\psi_i \tag{8a}$$

$$-2\psi_{i,x_1x_1} + V_i\psi_r + V_r\psi_i = E_r\psi_i + E_i\psi_r. \tag{8b}$$

In order to draw some meaningful conclusions about the real and imaginary parts of the eigenvalue E for a given potential $V(x)$ we solve equations (8a) and (8b) for E_r and E_i and obtain

$$E_r = -\frac{2}{\psi_r^2 + \psi_i^2}(\psi_r\psi_{i,x_1x_1} - \psi_i\psi_{r,x_1x_1}) + V_r \tag{9a}$$

$$E_i = -\frac{2}{\psi_r^2 + \psi_i^2}(\psi_r\psi_{r,x_1x_1} + \psi_i\psi_{i,x_1x_1}) + V_i. \tag{9b}$$

For the solution of (8) (or, for that matter, of (9)) we make an ansatz for $\psi(x)$ as

$$\psi(x_1, p_1) \equiv \psi_r + i\psi_i = \exp(g(x))$$
$$= \exp[g_r(x) + ig_i(x)] \qquad (10)$$

implying

$$\psi_r(x_1, p_2) = e^{g_r(x_1, p_2)} \cos g_i(x_1, p_2)$$
$$\psi_i(x_1, p_2) = e^{g_r(x_1, p_2)} \sin g_i(x_1, p_2). \qquad (11)$$

Using these forms for ψ_r and ψ_i equations (9a) and (9b) can be expressed as

$$g_{r,x_1 x_1} - (g_{i,x_1})^2 + (g_{r,x_1})^2 + \tfrac{1}{2}(E_r - V_r) = 0 \qquad (12a)$$

$$g_{i,x_1 x_1} + 2g_{i,x_1} g_{r,x_1} + \tfrac{1}{2}(E_i - V_i) = 0. \qquad (12b)$$

For a given complex potential $V(x)$ and the ansatz for g_r and g_i equations (12a) and (12b) can be rationalized to give E_i and E_r. Note that the ansatz for $g_r(x_1, p_2)$ and $g_i(x_1, p_2)$ should also conform to the analyticity property, namely

$$g_{r,x_1} = g_{i,p_2} \qquad g_{i,x_1} = -g_{r,p_2}. \qquad (13)$$

In what follows we apply the above results to obtain the solution of the ASE (3) for some typical one-dimensional potentials.

Case I. First, we consider the simplest example of a complex oscillator described by

$$V(x) = ax^2 \qquad (a, \text{ real}) \qquad (14)$$

or, after using (1), we have

$$V_r(x_1, p_2) = a(x_1^2 - p_2^2) \qquad V_i(x_1, p_2) = 2ax_1 p_2.$$

An ansatz for g_r and g_i, which conform to (13), is

$$g_r(x_1, p_2) = \tfrac{1}{2}\alpha(x_1^2 - p_2^2) + \beta x_1 p_2$$
$$g_i(x_1, p_2) = \tfrac{1}{2}\beta(-x_1^2 + p_2^2) + \alpha x_1 p_2 \qquad (15)$$

where α and β are real. Use of (15) in (12a) yields the expression

$$\alpha - (-\beta x_1 + \alpha p_2)^2 + (\alpha x_1 + \beta p_2)^2 + \tfrac{1}{2}E_r - \tfrac{1}{2}a(x_1^2 - p_2^2) = 0$$

which can be rationalized to give the following set of relations among the arbitrary constants α and β:

$$E_r = -2\alpha \qquad (16a)$$
$$\beta\alpha = 0 \qquad (16b)$$
$$-\beta^2 + \alpha^2 - \tfrac{1}{2}a = 0. \qquad (16c)$$

Equation (16b) suggests that either $\alpha = 0$ or $\beta = 0$. If $\alpha = 0$, then equation (16c) yields the imaginary value of β, which is in contrast with the ansatz (15). On the other hand, if $\beta = 0$, then $\alpha = \pm\sqrt{\frac{a}{2}}$ and equation (16a) leads to

$$E_r = +\sqrt{2a} \qquad (17)$$

for the negative sign in α.

Similarly, if we use the ansatz (15) in (12b) and rationalize the resultant expression then one obtains

$$E_i = 2\beta \qquad (18a)$$
$$-2\beta^2 + 2\alpha^2 - a = 0 \qquad (18b)$$
$$\alpha\beta = 0. \qquad (18c)$$

Note that the consistent values of α and β obtained from (18b) and (18c) are the same as those obtained from (16b) and (16c), leading to

$$E_i = 0. \tag{19}$$

Corresponding to these values of arbitrary constants α and β the eigenfunction $\psi(x_1, p_2)$ from (10) and (15) now becomes

$$\psi(x_1, p_2) = \exp\left[-\frac{1}{2}\sqrt{\frac{a}{2}}(x_1^2 - p_2^2 + 2\mathrm{i}x_1 p_2) \right]. \tag{20}$$

Case II. Next we look for the solution of (3) for the complex *harmonic plus inverse harmonic potential* of the form

$$V(x) = a_2 x^2 + \frac{a_1}{x^2} \qquad (a_2, a_1, \text{real}) \tag{21}$$

or, after using (1), we have

$$V_r(x_1, p_2) = a_2(x_1^2 - p_2^2) + \frac{a_1(x_1^2 - p_2^2)}{(x_1^2 + p_2^2)^2}$$

$$V_i(x_1, p_2) = 2a_2 x_1 p_2 - \frac{2a x_1 p_2}{(x_1^2 + p_2^2)^2}. \tag{22}$$

With regard to the ansatz for g_r and g_i in this case, while we seek guidance from our earlier work [13], the forms consistent with equation (13) can be chosen as

$$g_r(x_1, p_2) = \tfrac{1}{2}\alpha(x_1^2 - p_2^2) + \beta x_1 p_2 + \beta_1 \ln(x_1^2 + p_2^2) \tag{23a}$$

$$g_i(x_1, p_2) = \tfrac{1}{2}\beta(-x_1^2 + p_2^2) + \alpha x_1 p_2 + 2\beta_1 \tan^{-1}(p_2/x_1) \tag{23b}$$

where α and β are real. As before in case I, the rationalization of equations (12a) and (12b) for these forms of g_r and g_i and for the potential (22) leads [14] to the following acceptable values of α, β and β_1 in (23):

$$\alpha = \pm(a_2/2)^{1/2} \qquad \beta = 0 \qquad \beta_1 = \tfrac{1}{4}[1 \pm (1 + 2a_1)^{1/2}]. \tag{24}$$

The corresponding eigenvalue and eigenfunction are given by

$$E_r = -\sqrt{2a_2}\,[2 \pm (1 + 2a_1)^{1/2}] \qquad E_i = 0 \tag{24'}$$

$$\psi(x_1, p_2) = (x_1^2 + p_2^2)^{\beta_1} \mathrm{e}^{\pm(1/2)\sqrt{a_2/2}\{x_1^2 - p_2^2 + 2\mathrm{i}x_1 p_2\}} \mathrm{e}^{2\mathrm{i}\beta_1 \tan^{-1}(p_2/x_1)}$$

$$= (x_1^2 + p_2^2)^{\beta_1} \exp\left[\pm \tfrac{1}{2}(a_2/2)^{1/2}(x_1 + \mathrm{i}p_2)^2 + 2\mathrm{i}\beta_1 \tan^{-1}(p_2/x_1) \right] \tag{25}$$

where β_1 is given in (24). Note that in the study of the quantum mechanics of potential (21) the parameter a_1 is found [15] to take only some discrete values, namely $a_1 = \tfrac{1}{2}m(m - 1)$, where m is a positive integer, for the existence of a normalizable solution. Here, however, the normalization of $\psi(x_1, p_2)$ involves integration over the complex x plane and hence will make the situation different.

Case III. Lastly, we consider the case of a one-dimensional *complex Mörse* potential of the form

$$V(x) = V_0(\mathrm{e}^{-2ax} - 2\mathrm{e}^{-ax}) \qquad (V_0, a, \text{real}) \tag{26}$$

or, after using (1), one obtains

$$V_r(x_1, p_2) = V_0[\mathrm{e}^{-2ax_1}\cos 2ap_2 - 2\mathrm{e}^{-ax_1}\cos ap_2] \tag{27a}$$

$$V_i(x_1, p_2) = V_0[\mathrm{e}^{-2ax_1}\sin 2ap_2 + 2\mathrm{e}^{-ax_1}\sin ap_2]. \tag{27b}$$

In fact, such a one-dimensional Mörse potential (for real x) has been studied in the past [16] in the context of linear lattices. Its three-dimensional version (in terms of the radial coordinate alone which is tantamount to the study in one dimension) has been investigated recently [17] in the context of the so-called $\mathbb{P}\mathcal{J}$-symmetric quantum mechanics. In the latter case, however, a correspondence between a Mörse potential and the harmonic plus inverse harmonic potential of the type (21) is demonstrated with regard to the solution of the Schrödinger equation for these potentials. While we postpone such details to a future publication [14], we give here only the results for the potential (26) in the present framework.

In view of condition (13), an acceptable ansatz for g_r and g_i in this case turns out to be

$$g_r(x_1, p_2) = \beta_1 x_1 + \beta_2 e^{-\alpha x_1} \cos \alpha p_2 \qquad g_i(x_1, p_2) = \beta_1 p_2 - \beta_2 e^{-\alpha x_1} \sin \alpha p_2. \tag{28}$$

Further, the use of (28) and (27a) in (12a) yields the expression

$$\alpha^2 \beta_2 e^{-\alpha x_1} \cos \alpha p_2 + \alpha^2 \beta_2^2 e^{-2\alpha x_1} \cos 2\alpha p_2 + \beta_1^2 - 2\alpha \beta_1 \beta_2 e^{-\alpha x_1} \cos \alpha p_2$$
$$+ \tfrac{1}{2} E_r - \tfrac{1}{2} V_0 [e^{-2ax_1} \cos 2ap_2 - 2e^{-ax_1} \cos ap_2] = 0. \tag{29}$$

The rationalization of (29) suggests a natural choice for α as $\alpha = a$, and other relations as

$$E_r = -2\beta_1^2 \tag{30a}$$
$$a^2 \beta_2^2 - \tfrac{1}{2} V_0 = 0 \tag{30b}$$
$$a^2 \beta_2 - 2a\beta_1 \beta_2 + V_0 = 0. \tag{30c}$$

The unknown constants β_1 and β_2 in (28) can readily be obtained from (30b) and (30c) as

$$\beta_2 = \pm \sqrt{\frac{V_0}{2a^2}} \qquad \beta_1 = \frac{a}{2} \pm \sqrt{\frac{V_0}{2}}$$

and hence E_r from (30a) as

$$E_r = -(a/\sqrt{2} - \sqrt{V_0})^2 \tag{31a}$$

for the negative sign in β_2. Note that, while the rationalization of equation (12b) gives rise to the same set of values of α, β_2 and β_1 as above, it, however, yields

$$E_i = 0. \tag{31b}$$

The eigenfunction $\psi(x_1, p_2)$, in this case, turns out to be

$$\psi(x_1, p_2) \equiv \psi(x) = \exp \left[\left(\frac{a}{2} - \sqrt{\frac{V_0}{2}} \right) x - \sqrt{\frac{V_0}{2a^2}} e^{-\alpha x} \right]$$

which has the same mathematical structure as obtained [13] for the real x in (26).

Note that, while in all three cases discussed above the imaginary part of the eigenvalue, E_i, turns out to be zero, the real part E_r and the eigenfunction $\psi(x_1, p_2)$ are found to have the same mathematical structure as [13] for the case of real x. The complex extension of the variables through equation (1) for these cases seems trivial. But that is not true. The prescription proposed here for the solution of ASE (3) is quite general. No doubt it is applied here only to a few cases (cf cases I, II and III) when the potential $V(x)$ depends on x alone and involves only the real parameters but it is expected to work for a variety of potentials. In fact, one can apply the method to the cases when the potential V is a function of both x and x^* and also involves the complex parameters. In that case one opens a Pandora's box as far as the varieties of solvable cases of complex potentials and the associated eigenvalue spectra are concerned. For example, for the simplest choice of harmonic oscillator of case I other variants of $V(x, x^*)$ turn out to be (i) ax^{*2}, (ii) axx^*, (iii) $ax^*x + bx^2$, (iv) $ax^*x + bx^{*2}$, (v) $ax^{*2} + bx^2$ and (vi) $ax^{*2} + bx^2 + cx^*x$, where the parameters a, b, c, in general, are complex.

Moreover, all these cases can be handled using the same ansatz (15). Similarly, the variants for the potentials of cases II and III, in general for the complex parameters a_2, a_1, V_0 and a, are (i) $a_2 x^{*2} + a_1/x^{*2}$, (ii) $a_2 x x^* + a_1/x^{*2}$, (iii) $a_2 x^{*2} + a_1/x x^*$, (iv) $a_2 x x^* + a_1/x^2$, (v) $a_2 x^2 + a_1/x x^*$, (vi) $a_2 x^{*2} + a_1/x^2$, (vii) $a_2 x^2 + a_1/x^{*2}$, (viii) $a_2 x x^* + a_1/x x^*$, (ix) several other three-term constructions like $a_3 x^2 + a_2 x x^* + a_1/x^2$, etc, and (i) $V_0(e^{-2ax} - 2e^{ax^*})$, (ii) $V_0(e^{-2ax^*} - 2e^{ax})$ (iii) $V_0(e^{-2ax^*} - 2e^{ax^*})$, respectively. Note that the latter two categories of potentials can again be handled using the ansatze (23) and (28), respectively.

It may be mentioned that the nature of the eigenfunction and the associated complex eigenvalue spectrum for these potentials in general turns out to be very sensitive not only with reference to the variant of a particular class of $V(x, x^*)$ but also with respect to the nature of the underlying parameters. Just to give some glimpses of such features, we highlight the first variant of case I, namely when (a) $V(x, x^*) = ax^{*2}$ (a, real) and (b) $V(x) = ax^2$ ($a = a_r + ia_i$, complex). For the former case ASE (3) does not admit the solutions in the form (15) whereas for the latter the eigenvalues and eigenfunction are found [14] as

$$E_r = +(a_r + |a|)^{1/2} \qquad E_i = +(-a_r + |a|)^{1/2}$$

$$\psi(x_1, p_2) = \exp\left[\pm \frac{1}{4} \frac{(a + |a|)}{(a_r + |a|)^{1/2}} (x_1 + i p_2)^2\right].$$

Thus, one can visualize the sensitive dependence of the nature of eigenvalues and eigenfunctions on the type of complexity dealt with in $V(x, x^*)$, namely parametric and/or coordinate type. $\mathbb{P}\mathcal{J}$-symmetric potentials, studied extensively in recent years [3–8], deal mainly with the coordinate-type complexity and that too in a restricted domain of the complex phase space.

To summarize, we mention that a simple prescription for the solution of the ASE (3) for complex potentials is proposed in this letter. While further applications of the method are in progress, it is found to work well for a variety of complex potentials (namely polynomial, singular and exponential potentials) as far as the quasi-exact ground state solution of equation (3) is concerned.

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